THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 3030 Abstract Algebra 2024-25
Tutorial 6
17th October 2024

- Tutorial exercise would be uploaded to blackboard on Mondays provided that there is a tutorial class on that Thursday. You are not required to hand in the solution, but you are advised to try the problems before tutorial classes.
- Please send an email to echlam@math.cuhk.edu.hk if you have any questions.

1. A group action $\rho : G \times X \to X$ is called free if $g \cdot x = x$ for some x implies that $g = e$. (This is not the same as faithful as we require for all x for faithfulness.) Prove that an action is free if and only if the stabilizer G_x is trivial for all $x \in X$. Is the left regular action $g \cdot h = gh$ by G on itself a free action?

2. Let $V = \mathbb{R}^n$, then $GL(n, \mathbb{R})$ acts on V by $v \mapsto Av$ for any invertible matrix $A \in GL(n, \mathbb{R})$. Let $F(\mathbb{R}^n)$ be the set of all ordered basis of $\mathbb{R}^n$, then $GL(n, \mathbb{R})$ also acts on $F(\mathbb{R}^n)$ by left multiplication: $A \cdot [v_1, ..., v_n] \mapsto [Av_1, ..., Av_n]$. Show that this is a well-defined action that is free and transitive. (Here we use $[...]$ instead of $\{...\}$ to signify that it is an ordered n-tuple.)

3. Prove that the kernel of a group action ρ, where $\rho : G \to S_X$ is regarded as a homomorphism to the group of bijections on X, satisfies $\ker(\rho) = \bigcap_{x \in X} G_x$.

4. (a) For any group action, suppose that x and x' are in the same orbit, say $x' = g \cdot x$, then show that $G_{x'} = gG_x g^{-1}$.
   (b) Prove that for a finite group G, for any proper subgroup H, $G \neq \bigcup_{g \in G} gHg^{-1}$. (Hint: count how many distinct subgroups of the form gHg^{-1} are there.)
   (c) Let X be a finite G-set with at least two elements, suppose that the action is transitive, show that there is some $g \in G$ so that g has no fixed-point, i.e. there exists $g \in G$ so that $g \cdot x \neq x$ for all x.

   Some discussion. There is a one-line proof to this statement using Burnside's lemma, which counts the number of orbits using the cardinality of fixed point set of each $g \in G$. On the other hand, the statement still holds if we replace G with an infinite group. The proof following Q4 still works after modifying part (b) by a proper, finite index $H \leq G$. However, we can no longer use Burnside's lemma in the infinite case! Maybe even more intuitively, if G is an infinite group, there is no loss of group action information by passing from $\rho : G \to S_X$ to $G/\ker(\rho) \to S_X$, the latter becomes a finite group action.

5. Can you find a counter-example to Q4 if we drop the assumption that X is finite. Where did the proof given in Q4c fail in this case?

6. Let C be a conjugacy class of G with $|C| > 1$, show that there is an element $g \in G$ that does not commute with any element of C. (Hint: Pick a suitable action and apply Q4c.)

Theorem. (Burnside's lemma)
Let G be a finite group acting on finite set X, let X/G denote the set of orbits of the action and $X^g = \{x \in X : g \cdot x = x\}$ the set of fixed-points of $g \in G$, then

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

Proof. Consider the set $S = \{(g, x) \in G \times X : g \cdot x = x\} \subset G \times X$, it comes with two projections $\pi_1 : S \to G$ and $\pi_2 : S \to X$. This gives us two way of counting $|S|$, along the fibers of each map. Fix a $g \in G$, we have $\pi_1^{-1}(g) = \{(g, x) \in \{g\} \times X : g \cdot x = x\} = X^g$. On the other hand, fixing $x \in X$, we also have $\pi_2^{-1}(x) = \{(g, x) \in G \times \{x\} : g \cdot x = x\} = G_x$. Therefore, we can count the cardinality as follow,

$$|S| = \sum_{g \in G} |X^g| = \sum_{x \in X} |G_x|.$$

Dividing both sides by $|G|$, we obtain

$$\frac{1}{|G|} \sum_{g \in G} |X^g| = \sum_{x \in X} \frac{|G_x|}{|G|} = \sum_{x \in X} \frac{1}{|G \cdot x|} = \sum_{[y] \in X/G} \sum_{x \in [y]} \frac{1}{|G \cdot x|} = \sum_{[x] \in X/G} \frac{|G \cdot x|}{|G \cdot x|} = |X/G|.$$

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